J. Appl. Maths Mechs Vol. 55, No. 6, pp. 812–819, 1991
Printed in Great Britain.

0021-8928/91 $15.00+.00

VIBRATIONS OF A CYLINDER IN A CONCENTRIC VESSEL FILLED WITH A TWO-LAYER FLUID†

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(*Received* 12 *November* 1990)

A hybrid vibrational system containing a solid (a cylinder) with an elastic connection to a coaxial cylindrical cavity, completely filled with a heavy ideal stably stratified two-layer fluid, is considered. The combined self-consistent vibrations of the body and the fluid (of the internal waves) are studied. An explicit solution of the internal boundary value problem of an inhomogeneous liquid in an annular domain for a specified motion of the body is obtained. An integrodifferential equation of the Newton type is constructed on the basis of this. This equation describes the self-consistent oscillations of the cylinder. In the case of weak coupling of the interaction between the solid and the medium, an approximate solution is obtained using asymptotic methods and an analysis is carried out. Qualitative effects of the mutual effect of the motions of the cylinder and the fluid are found.

1. DESCRIPTION OF THE MECHANICAL MODEL

THE MOTIONS of the hybrid vibrational system which is shown schematically in Fig. 1 [(*a*) is a side or front view and (*b*) is a view from above] are investigated. There is a fixed cylindrical vessel B of radius b, the axis of symmetry z of which is directed along a vertical. An inertial coordinate system xyz is associated with the vessel. A cylinder A of radius a, $a<b$, and $b-a\sim b$ is arranged coaxially in it. The height of the vessel and the cylinder are the same and equal to h. It is assumed that the internal cylinder A can be moved in a plane-parallel manner without friction from the sides of the

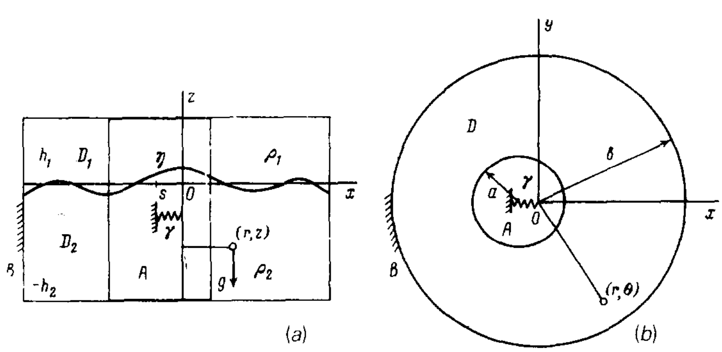

FIG. 1.

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 934–940, 1991.

end walls. The generating lines and the end walls of the cavity B and the moving body A are assumed to be absolutely rigid.

Vessel B is completely filled with an ideal two-layer fluid: the upper layer has a thickness $h_1>0$ and a density $\rho_1\geqslant 0$ while the lower layer has a thickness $h_2>0$ ($h_1\sim h_2\sim h$) and a density ρ_2 ($\rho_2>\rho_1$), that is, the fluid is stably stratified. A gravitational force with an acceleration equal to g acts vertically downwards on the particles of the fluid (Fig. 1). The internal body A is elastically coupled to the fixed vessel B so that their axes are coincident in the equilibrium position. The coupling is assumed to be linear with a coefficient γ. To be specific, we shall assume that the cylinder can be moved along the x-axis and the variable $s = s(t)$ characterizes the deviation of the cylinder from the equilibrium position $s = 0$. We shall assume that the unperturbed surface of separation lies in the $z = 0$ plane, that is, in the xy plane (as shown in Fig. 1).

The mechanical formulation of the problem involves the following. At the initial instant of time $t = 0$, the internal cylinder is assumed to be displaced along the x-axis by a small distance s^0, $|s^0|\ll b-a\sim b\sim a$. The surface of separation of the two layers of fluid is horizontal and the fluid is at rest. Cylinder A is released and is set into motion without an initial velocity under the action of an elastic restitution force and reaction on the part of the fluid. It is required to determine the motions of the cylinder A, the kinematic and dynamic characteristics of the fluid in the cavity B and also the internal waves which arise at the surface of separation.

2. FORMULATION OF THE BOUNDARY VALUE PROBLEM FOR THE FLUID FOR A SPECIFIED MOTION OF THE BODY

The wave motions of the fluid are considered in the linear approximation [1]. As a consequence of the axial (cylindrical) symmetry of the problem, it is convenient to introduce the corresponding coordinates (r, θ, z) which vary within the limits

$$a \leqslant r \leqslant b, \quad 0 \leqslant \theta < 2\pi, \quad -h_2 \leqslant z \leqslant h_1$$

We note that the fluid occupies the domain $D = B\backslash A$, where $\backslash$ is the operation of subtraction of the displaced set A from the fixed set B. This domain D is not simply connected. We denote by φ_1 and φ_2 the velocity potentials in each end layer $D_1(h_1\geqslant z>0)$ and $D_2(h_2\leqslant z<0)$, $D = D_1\cup D_2$. Let us represent them in the form

$$\begin{gathered}\varphi_{1,2} = \varphi_{1,2}(t, r, \theta, z) = \Phi_0(t, r, \theta) + \Phi_{1,2}(t, r, \theta, z) \\ \Phi_0 = \Phi_0(t, r, \theta) = a^2(r^2+b^2)(b^2-a^2)^{-1}r^{-1}s^{\bullet}(t)\cos\theta \end{gathered} \tag{2.1}$$

The first term Φ_0 is the velocity potential created by the cylinder A moving with a velocity $s^{\bullet} = s^{\bullet}(t)$. It possesses the following properties:

$$\begin{gathered}\Delta\Phi_0 = 0, \quad (r, \theta, z)\in D \\ -\Phi_{0r}'\big|_{r=a} = s^{\bullet}(t), \quad \Phi_{0r}'\big|_{r=b} = 0\end{gathered} \tag{2.2}$$

Here, the time t is treated as a parameter. It is more convenient to represent the Laplacian Δ in cylindrical coordinates (r, θ, z). The primes denote derivatives with respect to the spatial variables mentioned below and a derivative with respect to time t is indicated by a dot. The functions $\Phi_{1,2}$ also depend on the variable z. They are unknown and have to be determined; $s^{\bullet}(t) = ds(t)/dt$ is the velocity of motion of the cylinder A along the x-axis. the velocity potentials of the wave motion of the fluids must be determined as solutions of the Laplace equation in the corresponding domains

$$\Delta\Phi_{1,2} = 0, \quad (r, \theta, z) \in D_{1,2} \quad (D = D_1 \cup D_2) \tag{2.3}$$

which satisfy the dynamic and kinematic boundary conditions on the boundary of separation of the fluids $z = 0$ (the linear theory)

$$\begin{gathered} \rho_2 (\Phi_2^{\cdot} - g\eta)_{z=0} - \rho_1 (\Phi_1^{\cdot} - g\eta)_{z=0} = -(\rho_2 - \rho_1)\, \Phi_0^{\cdot} \\ \eta^{\cdot} = -\Phi_{1z}' \,|_{z=0} = -\Phi_{2z}' \,|_{z=0} \end{gathered} \tag{2.4}$$

Here $\eta = \eta(t, r, \theta)$ is an unknown function which determines the shape of the surface of separation of the fluids. Moreover, the condition of non-penetration of the fluids (of the impermeability of the walls) must be satisfied on the solid boundaries of the two cylinders, that is, the velocities of the fluid particles on the boundary of domain D are equal to zero:

$$\Phi_{1z}' \,|_{z=h_1} = \Phi_{1r}' \,|_{r=a,\, b} = \Phi_{2z}' \,|_{z=-h_2} = \Phi_{2r}' \,|_{r=a,\, b} = 0 \tag{2.5}$$

In accordance with the assumptions which were made in Sec. 1, the initial conditions for the potentials $\Phi_{1,2}$ have the form

$$\begin{gathered} \Phi_{1,2} (0, r, \theta, z) = 0, \quad (r, \theta, z) \in D_{1,2} \\ (\rho_2\Phi_2^{\cdot} - \rho_1\Phi_1^{\cdot})_{t=0,\, z=0} + (\rho_2 - \rho_1)\, \Phi_0^{\cdot} (0, r, \theta) = 0 \end{gathered} \tag{2.6}$$

The first condition also denotes that the velocities of the fluid particles are equal to zero while the second denotes that the shape of the surface of separate $\eta\,|_{t=0}$ is planar $[\eta(0, r, \theta) = 0]$ [see the first condition of (2.4)].

So, the boundary value problem (2.3)–(2.5) with the initial conditions (2.6) is posed for a known (specified) motion $s(t)$ of the cylinder A in the case of a stratified fluid. The solution of this equation $\Phi_{1,2}(s, r, \theta, z)$, $\eta(t, r, \theta)$ has to be determined. On this basis of this solution, we determine the dynamic rigidity conditions (in particular, the pressures on the surface of the moving cylinder A) and calculate the forces acting on the internal body due to the fluid. A Newton-type integrodifferential equation of the dynamics is then set up for a motion of the cylinder $s = s(t)$ taking account of all of the forces acting on the cylinder. The calculation of $s(t)$ enables one to obtain expressions for the required kinematic and dynamic characteristics of the fluid in closed form (see below).

In its conceptual plan, the formulation of the problem is close to the investigations in [2], which was concerned with the dynamics of a vessel which contains a two-layer fluid and is elastically coupled to a fixed base. In the investigation of the hydrodynamics of the problem however, the problem considered here differs considerably from that considered in [2]. It is much more difficult to solve, mainly due to the greater complexity of the domain D and the fact that it is not simply connected. We note that, in the general case, the shape of the cavity D is variable and depends on the motion of the internal body A. However, for small displacements s ($s \ll a$, $b - a$), this fact may be neglected in the linear approximation.

3. SOLUTION OF THE BOUNDARY VALUE PROBLEM

Using the method of separation of variables, we will represent the expressions for the unknown potentials $\Phi_{1,2}$ and the elevation η in the form of the series

$$\Phi_{1,2} = \cos\theta \sum_{n=1}^{\infty} \Theta_n^{(1,2)}(t)\,\psi_n(\lambda_n r)\,\mathrm{ch}\,(\lambda_n z_{1,2})$$

$$\eta = \cos\theta \sum_{n=1}^{\infty} \Theta_n(t)\,\psi_n(\lambda_n r) \quad (z_{1,2} = h_{1,2} \mp z) \tag{3.1}$$

The dependence on θ and z given in (3.1) follows from (2.3)–(2.5). The functions ψ_n and the numbers λ_n $(n \geqslant 1)$ are the solution of an eigenvalue problem and Sturm–Liouville functions of the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) + \left(\lambda^2 - \frac{1}{r^2}\right)\psi = 0, \quad a \leqslant r \leqslant b$$

$$\psi_r'\,|_{r=a,\,b} = 0 \tag{3.2}$$

The eigenfunctions $\psi_n = \psi_n(\lambda_n r)$ are expressed in terms of the first-order Bessel function J_1 and the Neumann function N_1 in the following manner:

$$\psi_n(\lambda_n r) = J_1(\lambda_n r)\,N_1'(\lambda_n a) - J_1'(\lambda_n a)\,N_1(\lambda_n r), \quad a \leqslant r \leqslant b \tag{3.3}$$

The eigenvalues λ_n $(n \geqslant 1)$ are determined as a two-parameter family of solutions of the transcendental equation

$$J_1'(\lambda b)\,N_1'(\lambda a) - J_1'(\lambda a)\,N_1'(\lambda b) = 0$$

$$\lambda_n = \lambda_n(a, b), \quad n = 1, 2, \ldots; \quad \lambda_n = O(n), \quad n \to \infty \tag{3.4}$$

By introducting the parameter $\xi = a\lambda$ (or $\xi = b\lambda$), we can reduce Eq. (3.4) to the form for which the roots $\xi_n = \xi_n(\mu)$, where $\mu = b/a > 1$ (or $\mu = a/b < 1$). Hence, the solution of the characteristic equation reduces to the construction of the single-parameter family of roots $\xi_n(\mu)$ for the values $\mu > 1$ (or $0 < \mu < 1$), which is more convenient from a computational point of view.

It can be shown by direct substitution that the functions $\Phi_{1,2}$ (3.1) satisfy the Laplace equation and the boundary conditions (2.5) which are specified on the absolutely rigid walls (on the boundary of the domain D). By using the dynamic and kinematic conditions (2.4) which are satisfied in the linear approximation on the boundary of separation between the two fluids, we obtain a system of first-order ordinary differential equations in the unknown Fourier coefficients $\Theta_n^{(1,2)}(t)$, $\Theta_n(t)$ $(n \geqslant 1)$:

$$\rho_2\,(\Theta_n^{(2)\cdot}\,\mathrm{ch}\,\lambda_n h_2 - g\Theta_n) - \rho_1\,(\Theta_n^{(1)\cdot}\,\mathrm{ch}\,\lambda_n h_1 - g\Theta_n) =$$

$$= -(\rho_2 - \rho_1)\,B_n s^{\cdot\cdot}(t), \quad t \geqslant 0 \tag{3.5}$$

$$\Theta_n^{\cdot} = \Theta_n^{(1)}\lambda_n\,\mathrm{sh}\,\lambda_n h_1 = -\Theta_n^{(2)}\lambda_n\,\mathrm{sh}\,\lambda_n h_2$$

$$B_n \equiv \frac{a^2}{(b^2 - a^2)\,\|\psi_n\|^2}\int_a^b (b^2 + r^2)\,\psi_n(\lambda_n r)\,dr, \quad \|\psi_n\|^2 = \int_a^b r\psi_n^2(\lambda_n r)\,dr$$

The estimates of the coefficients B_n and $\|\psi_n\|$ have the form $B_n \sim n^{-1}$, $\|\psi\| \sim 1$. By differentiating the first equation of (3.5) and eliminating the unknown Θ_n using the second, we obtain the relationships

$$\Theta_n^{(2)\cdot\cdot} + \omega_n^2\Theta_n^{(2)} = -(g\lambda_n\,\mathrm{sh}\,\lambda_n h_2)^{-1}\,B_n\omega_n^2 s^{\cdot\cdot\cdot}(t)$$

$$\Theta_n^{(1)} = -\Theta_n^{(2)}\,\frac{\mathrm{sh}\,\lambda_n h_2}{\mathrm{sh}\,\lambda_n h_1}, \quad \omega_n^2 = \frac{g\,(\rho_2 - \rho_1)\,\lambda_n\,\mathrm{th}\,\lambda_n h_1\,\mathrm{th}\,\lambda_n h_2}{\rho_1\,\mathrm{th}\,\lambda_n h_2 + \rho_2\,\mathrm{th}\,\lambda_n h_1} \tag{3.6}$$

for determining $\Theta_n^{(2)}$ and $\Theta_n^{(1)}$.

Here ω_n ($n = 1, 2, \ldots$) are the eigenfrequencies of the vibrations of the stratified fluid in the annular domain D. We note that $\omega_{n+1} - \omega_n \sim 1\sqrt{n} \to 0$ when $n \to \infty$, that is, the eigenfrequencies condense and they have a unique condensation point at infinity. Formula (3.6) is identical in form to those obtained in [2] in the case of a rectangular simply connected domain. For the time being it is assumed in Eq. (3.6) for $\Theta_n^{(2)}(t)$ that $s(t)$ is triply continuously differentiable. Let us write the general solution for $\Theta_n^{(2)}(t)$ in the form

$$\Theta_n^{(2)}(t) = \frac{-B_n\omega_n}{g\lambda_n \operatorname{sh}\lambda_n h_2}\left[\int_0^t \sin\omega_n(t-\tau)s^{\cdots}(\tau)\,d\tau + \alpha_n \sin\omega_n t + \beta_n \cos\omega_n t\right]$$

where α_n and β_n are arbitrary constants. On integrating by parts, we obtain an expression for the general solution $\theta_n^{(2)}$ in the form

$$\Theta_n^{(2)}(t) = \frac{-B_n\omega_n}{g\lambda_n \operatorname{sh}\lambda_n h_2}\Big[-s^{\cdot\cdot}(0)\sin\omega_n t + \omega_n s^{\cdot}(t) - \omega_n^2 \int_0^t \sin\omega_n(t-\tau)s^{\cdot}(\tau)\,d\tau + \alpha_n \sin\omega_n t + \beta_n\cos\omega_n t\Big] \quad (3.7)$$

which is more suitable for the subsequent investigation.

The function $\Theta_n^{(1)}$ is obtained from $\Theta_n^{(2)}$ (3.7) by multiplying by a constant coefficient in accordance with (3.6). On the basis of the initial conditions (2.6) for the potentials $\Phi_{1,2}$, we get the initial conditions for the coefficients $\theta_n^{(1,2)}$ and the derivatives

$$\Theta_n^{(1,2)}(0) = 0, \quad \rho_2\Theta_n^{(2)\cdot}(0)\operatorname{ch}\lambda_n h_2 - \rho_1\Theta_n^{(1)\cdot}(0)\operatorname{ch}\lambda_n h_1 + (\rho_2-\rho_1)B_n s^{\cdot\cdot}(0) = 0 \quad (3.8)$$

The equalities (3.8) are satisfied, if one puts $\alpha_n = s^{\cdot\cdot}(0)$, $\beta_n = 0$. Finally, we obtain the final expressions for the coefficients $\Theta_n^{(1,2)}$ ($n \geq 1$):

$$\Theta_n^{(1,2)}(t) = \frac{\pm B_n\omega_n}{g\lambda_n \operatorname{sh}\lambda_n h_{1,2}}\left[\omega_n s^{\cdot}(t) - \omega_n^2\int_0^t \sin\omega_n(t-\tau)s^{\cdot}(\tau)\,d\tau\right] \quad (3.9)$$

By taking account of expressions (3.9), we obtain the required representations for the potentials $\varphi_{1,2}$ (2.1). They are specified as linear integral operators of the unknown $s^{\cdot}(t)$. The variable $s(t)$ is to be determined [1, 2]. In order to set up the equations of motion of the inner cylinder A it is necessary to calculate the forces of reaction on the part of the vibrating fluid.

4. MOTION OF THE INTERNAL CYLINDER

In order to calculate the resulting forces of the pressures acting on the internal cylinder A, it is necessary to find the time derivatives $\varphi_1^{\cdot}$ and $\varphi_2^{\cdot}$ of the velocity potentials in each annular layer D_1, D_2. For this purpose, we differentiate the series (2.1), (3.1) and (3.9) and find the distribution of the normal pressures using the linearized Bernoulli integral

$$p = \rho\varphi^{\cdot} - \rho g y, \quad p = p_{1,2}, \quad \rho = \rho_{1,2}, \quad \varphi = \varphi_{1,2}, \quad (r, \theta, z) \in D_{1,2} \quad (4.1)$$

The projections of the forces of the normal pressures $\mathbf{R} = (X, Y, Z)^T$ (of the reactions) on the part of the fluid on the moving cylinder A are

$$X = -a \int_0^{2\pi} \int_{-h_2}^{h} \rho\varphi^{\cdot} |_{r=a} \cos\theta \, d\theta \, dz =$$

$$= -\left[\pi a^2 \frac{a^2+b^2}{b^2-a^2} (\rho_1 h_1 + \rho_2 h_2) - \frac{\pi a}{g} (\rho_2 - \rho_1) \sum_{n=1}^{\infty} \frac{B_n \omega_n^2}{\lambda_n^2} \psi_n (\lambda_n a) \right] s^{\cdot\cdot}(t) -$$

$$- \frac{\pi a}{g} (\rho_2 - \rho_1) \sum_{n=1}^{\infty} \frac{B_n \omega_n^4}{\lambda_n^2} \psi_n (\lambda_n a) \int_0^t \cos \omega_n (t-\tau) s^{\cdot}(\tau) \, d\tau \qquad (4.2)$$

$$Y = Z \equiv 0 \quad (\rho = \rho(z))$$

$$(Z_\Sigma = Z + Z_A - M_A g + Z_* = 0)$$

Here, account has been taken of the fact that integration of the hydrostatic pressure forces yields a null projection on the Ox and Oy axes. Furthermore, the quantity Z_A in (4.2) is the repulsive Archimedean force, $M_A g$ is the force of the weight of the internal cylinder and Z_* is the force of the normal reaction on its ends on the part of the cavity. Apart from this, a linear elastic force with a stiffness γ acts on the cylinder A along the Ox axis. As a result, we get a Newton-type integrodifferential equation of motion with the initial conditions

$$M_A s^{\cdot\cdot} = -\gamma s + X(s^{\cdot\cdot}, [s^{\cdot}]), \quad s(0) = s^0, \; s^{\cdot}(0) = 0 \qquad (4.3)$$

It follows from an analysis of expression (4.2) of the force X in (4.3) that the force X contains an inertial addition which is proportional to $s^{\cdot\cdot}(t)$ and is due to the coupled masses (of a stationary and dynamic character) and, also, a term which is an integral of $s^{\cdot}(t)$ which takes account of the interaction between the waves and the cylinder. This interaction is described by an integral operator of the Volterra type. It obviously follows from (4.2) and (4.3) that, when $\varphi = 0$ the displacement $s(t) \equiv s^0$, $t \geqslant 0$ ($s^{\cdot}(t) \equiv 0$).

Let us now transform the Cauchy problem (4.3) to a form which is more convenient for analysis:

$$s^{\cdot\cdot} + \Omega^2 s = -\varepsilon \int_0^t K(t-\tau) s^{\cdot}(\tau) \, d\tau, \quad s(0) = s^0, \quad s^{\cdot}(0) = 0$$

$$\Omega^2 = \frac{\gamma}{M^*}, \quad M^* = M_A + \pi a^2 \frac{a^2+b^2}{b^2-a^2} (\rho_1 h_1 + \rho_2 h_2) + 2 \frac{\rho_2 - \rho_1}{g} \sum_{n=1}^{\infty} \frac{B_n \omega_n^2}{\lambda_n^3}$$

$$\varepsilon K(t) = \frac{\rho_2 - \rho_1}{M^* g} \sum_{n=1}^{\infty} \frac{B_n \omega_n^4}{\lambda_n^3} \cos \omega_n t \equiv \varepsilon \sum_{n=1}^{\infty} k_n \cos \omega_n t \qquad (4.4)$$

Here Ω is the characteristic frequency of the oscillations of the cylinder A, M^* is its effective mass, $\varepsilon > 0$ is a numerical parameter, for example, $\varepsilon = 2\pi a^2 h(\rho_2 - \rho_1)/M^*$, and $K(t-\tau)$ is the difference kernel of the integral operator. The corresponding trigonometric series for $K(t)$ is absolutely and uniformly convergent for all $t \geqslant 0$ since, according to (3.5) and (3.6) the estimates $B_n \sim \lambda_n^{-1} \sim n^{-1}$ and $\omega_n \sim \lambda_n^{1/2} \sim \sqrt{n}$ hold for B_n and ω_n when $n \to \infty$ and the coefficients $k_n \sim n^{-2}$. We note that problem (4.4) is identical to that obtained in [2] with respect to its external form.

An exact solution of the Cauchy problem (4.4) can be constructed using operator methods. However, it turns out to be inconvenient for qualitative analysis [2]. In the general case, numerical methods can be used to calculate the motions in a certain bounded time interval. If the interaction between the cylinder and the internal waves is small, that is, the numerical parameter ε is sufficiently small ($0<\varepsilon\leqslant 1$) then, for the qualitative analysis of system (4.4), use can be made of the asymptotic average methods developed in [2, 3]. The mathematical basis of these methods is given in [4]. We will now present the results of this analysis.

Let us first consider the case of an "internal resonance" where, for values of $n = N = 1, 2, \ldots$ which are not too large, the closeness of the frequency ω_N and Ω holds: $|\Omega - \omega_N| = O(\sqrt{\varepsilon})$. Then, according to [2], we have

$$\begin{gathered} s_* = s^0 \cos \sqrt{\varepsilon k_N} t \cos \Omega t, \quad | s - s_* | = O(\sqrt{\varepsilon}) \\ v_* = -s^0 \Omega \cos \sqrt{\varepsilon k_N} t \sin \Omega t, \quad | s^{\cdot} - v_* | = O(\sqrt{\varepsilon}) \\ 0 \leqslant t \leqslant c / \sqrt{\varepsilon}, \quad c = \text{const} \end{gathered} \tag{4.5}$$

It follows from (4.5) that "beating" occurs in the system, that is, the amplitude S of the oscillations of the cylinder A varies periodically with a low frequency $\sim\sqrt{\varepsilon}$ since we have $S = s^0 \cos \sqrt{(\varepsilon k_N)} t$. There is a slow exchange of energy between the cylinder and the fluid and, at each stage, there is a transfer of the energy E of the oscillations of the cylinder to the internal waves of the Nth mode and a change in its total energy up to values of $O(\sqrt{\varepsilon})$, that is, practically to zero (for small ε): $E = E^0 \cos^2 \sqrt{(\varepsilon k_N)} t + O(\sqrt{\varepsilon})$ which is followed again by the build up of an oscillation of the cylinder by means of the action of the Nth mode of the vibrations of the internal wave. The mathematical details of the analysis are contained in [2].

There is considerable interest in the theoretical and applied aspects of the investigation of the motions in the case of very large values of t, $t \gg 1/\sqrt{\varepsilon}$ such as $t \sim 1/\varepsilon$ or $t \to \infty$, for example. It is clear from physical considerations that the picture of the beats will be blurred; allowance for dissipation naturally leads to a decay in the oscillations of the cylinder and the internal waves of the fluid.

Let us now consider the case when there is no internal resonance, that is, when $|\Omega - \omega_n| = O(1)$ for $n = 1, 2, \ldots$ Then, in accordance with the results in [3], we have:

$$\begin{gathered} s^* = s^0 \cos \nu t, \quad | s - s^* | = O(\varepsilon) \\ v^* = -s^0 \Omega \sin \nu t, \quad | s^{\cdot} - v^* | = O(\varepsilon) \\ \nu = \nu(\varepsilon) = \Omega(1 + \varepsilon \Lambda / 4), \quad \Lambda = \sum_{n=1}^{\infty} \frac{k_n}{\Omega^2 - \omega_n^2} \\ \Lambda \gtrless 0; \quad 0 \leqslant t \leqslant c / \varepsilon, \quad c = \text{const} \end{gathered} \tag{4.6}$$

It follows from (4.6) that, when there is no resonance for $t \sim 1/\varepsilon$, there is practically no interaction between the oscillations of the cylinder and the fluid (there are no beats and the energy of the cylinder is conserved for small $\varepsilon > 0$). However, the oscillations of the cylinder have a frequency which is shifted by an amount $O(\varepsilon)$. The addition to the frequency may be greater or less than zero, which is an extremely interesting phenomenon. As previously, in the case when there is no resonance, the question as to the global readjustment of the motion for $t \gg 1/\varepsilon$ ($t \to \infty$) is of interest.

The results obtained above hold for arbitrary values of the parameters a, b, h_1, h_2 and for other changes in them in the above-mentioned domains. There is considerable interest in the limiting cases such as, for example, when $a/b \ll 1$ ($a \to 0$, $b \sim 1$) or $b/a \gg 1$ ($a \sim 1$, $b \to \infty$), etc. The first of the cases mentioned above can obviously be investigated on the basis of the constructions in Secs 2–4. The case when $b \to \infty$ is special since the frequency spectrum of the internal waves is condensed, that is, the eigenvalues λ_n become continuous in the limit. This is an important fact which requires a separate treatment.

If the solution $s(t)$ of the Cauchy problem (4.4) is constructed in an analytical or numerical form, all the kinematic and dynamic characteristics of the motions of the fluid are determined using the constructions in Secs 2 and 3 and formula (4.1).

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Translated by E.L.S.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 819–827, 1991
Printed in Great Britain.

0021–8928/91 $15.00+.00

THE THERMAL WAKE OF A STREAMLINED BODY†

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(*Received* 31 *August* 1990)

The stationary problem of the thermal wake behind a body around which there is a flow of a viscous incompressible fluid is considered within the framework of the full heat-conduction equation. It is assumed that the solution of the corresponding hydrodynamic problem is known. In the case of the hydrodynamic problem, theorems of existence [1, 2] and uniqueness [1] have been proved and the leading term of the expansion [1, 3] at an infinitely remote point has been obtained together with estimates of the remaining terms [1, 4]. Work mainly carried out within the framework of the boundary layer approximation [5] is concerned with the solution of the thermal problem.

1. THE SOLUTION of the hydrodynamic problem can be represented in the form

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}_0 + \mathbf{w}(\mathbf{x}), \quad \mathbf{w} = O(1/r), \quad r = |\mathbf{x}| \tag{1.1}$$

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 941–948, 1991.